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Krull–Schmidt reduction of principal bundles in arbitrary characteristic

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Abstract

Let M be an irreducible projective variety defined over an algebraically closed field k, and let E_G be a principal G-bundle over M, where G is a connected reductive linear algebraic group defined over k. We show that for E_G there is a naturally associated conjugacy class of Levi subgroups of G. Given a Levi subgroup H in this conjugacy class, the principal G-bundle E_G admits a reduction of structure group to H. Furthermore, this reduction is unique up to an automorphism of E_G.

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1. Introduction

Let M be a projective variety defined over an arbitrary field k. A vector bundle V over M is called indecomposable if V is not isomorphic to a direct sum of two vector bundles of positive rank. Using induction on rank it follows that any vector bundle over M is isomorphic

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to a direct sum of indecomposable vector bundles. A theorem due to Atiyah says that if

$$V = \bigoplus_{i=1}^{m} V_i$$

and

$$V = \bigoplus_{i=1}^{n} V_i'$$

are two decompositions of V into direct sum of indecomposable vector bundles, then $m = n$, and furthermore, there is a permutation σ of $\{1, \ldots, m\}$ such that V_i' is isomorphic to $V_{\sigma(i)}$ for all $i \in [1, m]$; see [1].

In [2], this was generalized to principal bundles but under the assumptions that the base field k is algebraically closed and its characteristic is zero.

Let k be an algebraically closed field. Let G be a connected reductive linear algebraic group defined over the algebraically closed field k. By a Levi subgroup of G we will mean a connected reductive algebraic subgroup H of some parabolic subgroup P of G such that H projects isomorphically onto the quotient of P by its unipotent radical $R_{\mathrm{u}}(P)$.

Given a principal G-bundle E_G over M, in [2] a natural reduction of structure group $E_H \subset E_G$ to a certain Levi subgroup $H \subset G$ associated to E_G is constructed under the assumption that the characteristic of k is zero. For $G = \mathrm{GL}(n, k)$ this reduction gives the above mentioned result of [1]. It should be emphasized that the result of [1] is valid for any field not necessarily algebraically closed.

The above-mentioned reduction of structure group $E_H \subset E_G$ depends on the choice of a maximal torus in the automorphism group of E_G. If E_{H_1} is another such reduction with $H_1 \subset G$ in the conjugacy class, then there is an element $g \in G$ and an automorphism τ of E_G such that $H_1 = gHg^{-1}$ and $E_{H_1}g = \tau(E_H) \subset E_G$.

Although some of the intermediate steps in [2] are not valid if k is of positive characteristic, it is natural to ask if the final reduction of structure group constructed in [2] still exists if k is of positive characteristic.

Our aim here is to give a variation of the construction of the reduction of structure group (done in [2]) which is valid for any characteristic. Consequently, the main result of [2] remains valid for algebraically closed fields of arbitrary characteristic.

2. Construction of reduction

Let G be a connected reductive linear algebraic group defined over an algebraically closed field k. Let M be an irreducible projective variety defined over the field k.

Take a principal G-bundle E_G over M. Consider the reduced group defined by all automorphisms of the G-bundle E_G. We recall that an automorphism of E_G is an isomorphism of the underlying variety that commutes with the action of G and induces the identity map of M. This reduced group will be denoted by $\mathrm{Aut}_{\mathrm{r}}(E_G)$. In other words, $\mathrm{Aut}_{\mathrm{r}}(E_G)$ is the reduced automorphism group scheme of E_G.

It is well-known that $\mathrm{Aut}_{\mathrm{r}}(E_G)$ is an algebraic group. However, we will put down below a proof of it.

Fix a finite-dimensional faithful left G-module V. Let $E_V = E_G(V)$ be the vector bundle over M associated to E_G for the G-module V. We recall that E_V is the quotient of $E_G \times V$ by the following action of G: the action of any $g \in G$ sends a point $(z, v) \in (E_G, V)$ to $(zg, g^{-1}v)$. Any automorphism of the principal G-bundle E_G induces an automorphism of any associated bundle. In particular, any automorphism of E_G induces an automorphism of the vector bundle E_V.

We will briefly recall the construction of this induced automorphism. For any automorphism γ of the principal G-bundle E_G, consider the automorphism of $E_G \times V$ defined by $(z, v) \longmapsto (\gamma(z), v)$. This automorphism of $E_G \times V$ descends to an automorphism of the quotient space $E_V = (E_G \times V)/G$.

Since the G-module V is faithful, an automorphism of the principal G-bundle E_G is determined by the induced automorphism of E_V. In other words, for any two distinct automorphisms of the principal G-bundle E_G, the corresponding induced automorphisms of E_V remain distinct.

Consider the reduced subscheme of the affine space $H^0(M, \mathrm{End}(E_V))$ given by all automorphisms of E_V that arise as induced automorphisms (induced by automorphisms of the principal G-bundle E_G). The earlier defined group $\mathrm{Aut}_{\mathrm{r}}(E_G)$ is identified with this reduced subscheme of $H^0(M, \mathrm{End}(E_V))$.

Let $\mathrm{Ad}(E_G) := E_G(G)$ be the group scheme over M associated to E_G. We recall that $\mathrm{Ad}(E_G)$ is the quotient of $E_G \times G$ for the following action of G: the action of any $g \in G$ sends a point $(z, g') \in (E_G, G)$ to $(zg, g^{-1}g'g)$. This group scheme $\mathrm{Ad}(E_G)$ is called the *adjoint bundle* (or the *gauge bundle*) of E_G. It is easy to see that any automorphism of the principal G-bundle E_G gives a canonically defined section of the fiber bundle $\mathrm{Ad}(E_G)$ over M, and conversely, any section of $\mathrm{Ad}(E_G)$ gives an automorphism of the principal G-bundle E_G. Indeed, this follows from the fact that any automorphism of the variety G that commutes with all the right translations, of G by itself, must be a left translation of G. Therefore, the group $\mathrm{Aut}_{\mathrm{r}}(E_G)$ is identified with the group defined by the sections of the group scheme $\mathrm{Ad}(E_G)$ over M.

Let

$$\psi: M \times \mathrm{Aut}_{\mathrm{r}}(E_G) \longrightarrow \mathrm{Ad}(E_G)$$

be the morphism defined by

$$(x, \gamma) \longmapsto \widehat{\gamma}(x), \tag{1}$$

where $\widehat{\gamma}$ is the section of the fiber bundle $\mathrm{Ad}(E_G)$ over M corresponding to the automorphism γ of E_G. For any closed point $x \in M$, let

$$\psi_x: \mathrm{Aut}_{\mathrm{r}}(E_G) \longrightarrow \mathrm{Ad}(E_G)_x \tag{2}$$

be the restriction to $\{x\} \times \mathrm{Aut}_{\mathrm{r}}(E_G) \subset M \times \mathrm{Aut}_{\mathrm{r}}(E_G)$ of the above morphism ψ.

Take any semisimple element

$$\gamma \in \mathrm{Aut}_{\mathrm{r}}(E_G). \tag{3}$$

So, for any closed point $x \in M$, the element

$$\psi_x(\gamma) \in \mathrm{Ad}(E_G)_x$$

is semisimple, where ψ_x is defined in (2). Each closed point z of the fiber $(E_G)_x$ gives an isomorphism of G with $\mathrm{Ad}(E_G)_x$ which is defined by

$$g \longmapsto (z, g),$$

$g \in G$ (recall that $\mathrm{Ad}(E_G)$ is a quotient of $E_G \times G$). Since these isomorphisms differ by inner automorphisms of G, we have an identification of the group $\mathrm{Ad}(E_G)_x$ with G up to an inner automorphism of G. Consequently, $\psi_x(\gamma)$ gives a well-defined conjugacy class in G. We will show that this conjugacy class in G does not depend on the choice of the point x (it depends only on γ).

Take any function f on G which is invariant under inner automorphisms of G. The function on $E_G \times G$ defined by $(z, g) \longmapsto f(g)$ descends to a function on the quotient space $\mathrm{Ad}(E_G)$. This descended function on $\mathrm{Ad}(E_G)$ will be denoted by $\widehat{f}$. Therefore, $\widehat{f} \circ \widehat{\gamma}$ is a function on M, where $\widehat{\gamma}$ is the section of $\mathrm{Ad}(E_G)$, as in (1), corresponding to γ. Since M is complete and connected, $\widehat{f} \circ \widehat{\gamma}$ is a constant function. Since $\psi_x(\gamma)$ is semisimple, this implies that the conjugacy class of $\psi_x(\gamma)$ is independent of x. Indeed, this follows from the fact that the space of invariant functions on G separate conjugacy classes of semisimple elements.

Fix an element

$$g_0 \in G \tag{4}$$

in the conjugacy class of G defined by $\psi_x(\gamma)$.

Let

$$q: E_G \times G \longrightarrow \mathrm{Ad}(E_G)$$

be the quotient map. Consider the inverse image

$$\mathscr{Z} = q^{-1}(\widehat{\gamma}(M)) \subset E_G \times G \tag{5}$$

which is an irreducible subvariety, where q is the above projection, and $\widehat{\gamma}$ is as in (1).

Let p_E (respectively, p_G) be the projection of $E_G \times G$ to E_G (respectively, G), and let p be the natural projection of $E_G \times G$ to M.

On $\mathscr{Z}$ defined in (5) we have the maps

$$E_G \xleftarrow{p_E|_{\mathscr{Z}}} \mathscr{Z} \xrightarrow{(p \times p_G)|_{\mathscr{Z}}} M \times \mathrm{Orb}(g_0) \subset M \times G, \tag{6}$$

where $\mathrm{Orb}(g_0)$ is the image of the map $G \longrightarrow G$ defined by $g \longmapsto g^{-1} g_0 g$ with g_0 being the fixed element in (4), or in other words, $\mathrm{Orb}(g_0)$ is the orbit of g_0 under the adjoint action of G on itself.

Let

$$Y_{g_0} := M \times \{g_0\} \subset M \times \mathrm{Orb}(g_0) \tag{7}$$

be the subvariety.

Lemma 2.1. *The inverse image* $((p \times p_G)|_{\mathscr{X}})^{-1}(Y_{g_0}) \subset \mathscr{X}$ *is reduced, where the map* $(p \times p_G)|_{\mathscr{X}}$ *is as in* (6), *and* Y_{g_0} *is defined in* (7).

Proof. Consider the map $f : G \longrightarrow G$ defined by $g \longmapsto g^{-1} g_0 g$, where $g_0 \in G$ is as in (4). It suffices to show that $f^{-1}(g_0) \subset G$ is reduced. But this follows from the fact that g_0 is semisimple. For, consider the adjoint action of g_0 on the Lie algebra $\mathfrak{g}$ of G; let $\eta \in \mathrm{Aut}(\mathfrak{g})$ denote this automorphism given by g_0. The tangent subspace $T_{g_0} \mathrm{Orb}(g_0) \subset T_{g_0} G$ is the direct sum of the eigenspaces of η for all the eigenvalues λ with $\lambda \neq 1$, and the tangent bundle of the fiber $f^{-1}(g_0)$ is identified with the trivial vector bundle over $f^{-1}(g_0)$ whose fiber is the eigenspace of η for the eigenvalue 1. Therefore, the subscheme $f^{-1}(g_0) \subset G$ is reduced. $\square$

Let

$$\mathscr{R} := f^{-1}(g_0) = C_G(g_0) \tag{8}$$

be the closed subgroup defined in the proof of Lemma 2.1. Here $C_G(g_0)$ denotes the centralizer of g_0 in G.

Proposition 2.2. *The image*

$$(p_E|_{\mathscr{X}})(((p \times p_G)|_{\mathscr{X}})^{-1}(Y_{g_0})) \subset E_G$$

is a reduction of structure group of E_G *to the subgroup* $\mathscr{R} \subset G$ *defined in* (8), *where* $p_E|_{\mathscr{X}}$ *and* $((p \times p_G)|_{\mathscr{X}})$ *are the maps in* (6), *and* Y_{g_0} *is defined in* (7).

Proof. Since $((p \times p_G)|_{\mathscr{X}})^{-1}(Y_{g_0})$ is reduced (see Lemma 2.1), its image

$$(p_E|_{\mathscr{X}})(((p \times p_G)|_{\mathscr{X}})^{-1}(Y_{g_0})) \subset E_G$$

is a reduced subvariety of E_G. The restriction to $(p_E|_{\mathscr{X}})(((p \times p_G)|_{\mathscr{X}})^{-1}(Y_{g_0}))$ of the natural projection of E_G to M is clearly smooth and surjective. Now it is straight-forward to check that the group $\mathscr{R}$ (defined in (8)) acts transitively on the fibers of the projection of $(p_E|_{\mathscr{X}})(((p \times p_G)|_{\mathscr{X}})^{-1}(Y_{g_0}))$ to M (see [2] for the details). This completes the proof of the proposition. $\square$

Remark 2.3. Let $E_{\mathscr{R}} \subset E_G$ be the reduction of structure group of E_G to $\mathscr{R}$ constructed in Proposition 2.2. If we replace g_0 by $g_0' = \alpha^{-1} g_0 \alpha$, where $\alpha \in G$, then the centralizer $\mathscr{R} \subset G$ is replaced by $\mathscr{R}' = \alpha^{-1} \mathscr{R} \alpha$, and the reduction of structure group $E_{\mathscr{R}}$ gets replaced by $E_{\mathscr{R}'} := E_{\mathscr{R}} \alpha \subset E_G$ (which is a reduction of structure group of E_G to $\mathscr{R}'$).

Let $\mathscr{T} \subset \mathrm{Aut}_{\mathrm{r}}(E_G)^0$ be a maximal torus, where $\mathrm{Aut}_{\mathrm{r}}(E_G)^0 \subset \mathrm{Aut}_{\mathrm{r}}(E_G)$ is the connected component of the group $\mathrm{Aut}_{\mathrm{r}}(E_G)$ containing the identity element. Fix an element $\gamma \in \mathscr{T}$ such that the centralizer of the element $\psi_x(\gamma)$ in $\mathrm{Ad}(E_G)_x$ coincides with the centralizer of $\psi_x(\mathscr{T})$, where ψ_x is the evaluation map in (2). The following Lemma 2.4 shows that such an element γ exists. (Although this lemma is well-known, we have included a proof of it for the convenience of the reader.)

Lemma 2.4. *Let H be a connected linear algebraic group defined over k and $T \subset H$ a torus. Then there exists an element $\xi \in T$ such that $C_G(\xi) = C_G(T)$, where $C_G(\xi)$ and $C_G(T)$ are the centralizers in G of ξ and T, respectively.*

Proof. If $G_1 \subset G_2$ are algebraic groups and $G' \subset G_1$ a commutative subgroup, then clearly $C_{G_1}(G')_{\mathrm{red}} = (C_{G_2}(G') \cap G_1)_{\mathrm{red}}$. Also note that if G' is a torus, then both $C_{G_1}(G')$ and $C_{G_2}(G') \cap G_1$ are reduced. In view of these observations it suffices to prove the lemma for $H = \mathrm{GL}(n,k)$.

After conjugation we can assume that the torus $T \subset \mathrm{GL}(n,k)$ is of the form

$$T = \bigoplus_{i=1}^{m} \mathbb{G}_m \cdot \mathrm{Id}_{V_i}$$

with $k^n = \bigoplus_{i=1}^{m} V_i$. Now take an element

$$\sum_{i=1}^{m} \lambda_i \cdot \mathrm{Id}_{V_i} =: \xi \in T$$

with the scalars λ_i pairwise distinct. Then clearly

$$C_{\mathrm{GL}(n,k)}(\xi) = \prod_{i=1}^{m} \mathrm{GL}(V_i) = C_{\mathrm{GL}(n,k)}(T).$$

This completes the proof of the lemma. $\square$

Henceforth we impose the following two conditions on γ:

- $\gamma \in \mathscr{T}$, and
- the centralizer of the element $\psi_x(\gamma)$ in $\mathrm{Ad}(E_G)_x$ coincides with the centralizer of the subgroup $\psi_x(\mathscr{T}) \subset \mathrm{Ad}(E_G)_x$,

where $\mathscr{T} \subset \mathrm{Aut_r}(E_G)^0$ is the fixed maximal torus.

The earlier observation that the conjugacy class in G defined by $\psi_x(\gamma)$ is independent of x implies that the above condition that the centralizer of $\psi_x(\gamma) \in \mathrm{Ad}(E_G)_x$ coincides with the centralizer of $\psi_x(\mathscr{T})$ actually does not depend on the choice of the point $x \in M$.

Theorem 2.5. *The reduction of structure group $E_{\mathscr{R}} \subset E_G$ to the subgroup $\mathscr{R} \subset G$ constructed in Proposition 2.2 is independent of the choice of the element $\gamma \in \mathscr{T}$ in the sense that if we replace γ by any $\gamma_1 \in \mathscr{T}$ satisfying the above conditions, then there is an element $g' \in G$ such that $\mathscr{R}$ is replaced by $(g')^{-1}\mathscr{R}g'$, and the reduction $E_{\mathscr{R}}$ gets replaced by $E_{\mathscr{R}}g'$.*

Furthermore, the reduction of structure group $E_{\mathscr{R}}$ to $\mathscr{R}$ constructed in Proposition 2.2 is independent of the choice of the maximal torus $\mathscr{T} \subset \mathrm{Aut_r}(E_G)^0$ in the sense that for another reduction of structure group $E_{\mathscr{R}'} \subset E_G$ to $\mathscr{R}' \subset G$ corresponding to another *choice of the maximal torus of* $\mathrm{Aut_r}(E_G)^0$, *there is an automorphism* $\tau \in \mathrm{Aut_r}(E_G)$ *and an element* $\alpha \in G$ *such that* $\mathscr{R}' = \alpha^{-1}\mathscr{R}\alpha$ *as well as* $\tau(E_{\mathscr{R}'}) = E_{\mathscr{R}}\alpha \subset E_G$.

Proof. Take an element $\gamma_1 \in \mathscr{T}$ such that the centralizer of the element $\psi_x(\gamma_1)$ in $\mathrm{Ad}(E_G)_x$ coincides with the centralizer of $\psi_x(\mathscr{T})$. Fix an element $g_0' \in G$ in the conjugacy class in G defined by $\psi_x(\gamma_1)$. Therefore, g_0' is obtained by replacing γ with γ_1 in the construction of g_0 in (4). Take $g' \in G$ such that the centralizer of g_0' in G coincides with $(g')^{-1}\mathscr{R}g'$. Since the centralizers of $\psi_x(\gamma)$ and $\psi_x(\gamma_1)$ coincide, such an element g' exists. This g' satisfies both the conditions in the first assertion in the theorem.

The second part follows using Remark 2.3 together with the fact that any two maximal tori in $\mathrm{Aut_r}(E_G)^0$ differ by an inner automorphism of $\mathrm{Aut_r}(E_G)^0$. $\square$

Remark 2.6. The centralizer of a torus in G is a Levi subgroup [3, p. 26, Proposition 1.22]. Therefore, the group $\mathscr{R}$ in Theorem 2.5 is a Levi subgroup of G. The reduction of structure group constructed in Theorem 2.5 clearly coincides with the one that was constructed in [2] under the extra assumption that the characteristic of k is zero.

Consider the linear group $\mathrm{GL}(n,k)$. A parabolic subgroup of it corresponds to a filtration of the vector space $k^{\oplus n}$, and a Levi subgroup of a parabolic subgroup of $\mathrm{GL}(n,k)$ corresponds to a complete decomposition of the filtration of $k^{\oplus n}$ defined by the parabolic subgroup. Therefore, if we set $G = \mathrm{GL}(n,k)$, then a reduction of structure group of E_G to a Levi subgroup of $\mathrm{GL}(n,k)$ corresponds to a decomposition of the vector bundle E of rank n over M associated to E_G by the standard representation of $\mathrm{GL}(n,k)$. From the conditions on γ that

- $\gamma \in \mathscr{T}$, and
- the centralizer of the element $\psi_x(\gamma)$ in $\mathrm{Ad}(E_G)_x$ coincides with the centralizer of $\psi_x(\mathscr{T})$

it follows that each direct summand of the vector bundle E for the decomposition corresponding to the reduction of structure group $E_{\mathscr{R}} \subset E_G$ constructed in Theorem 2.5 is in fact indecomposable. Since any endomorphism of an indecomposable vector bundle V over M is of the form $\lambda \cdot \mathrm{Id}_V + N$, where $\lambda \in k$ and N is a nilpotent endomorphism of V [1], if we have a decomposition of the vector bundle E into a direct sum of indecomposable vector bundles, then the corresponding reduction of structure group of E_G to a Levi subgroup coincides with the one given in Theorem 2.5 for some choice of $\mathscr{T}$. Consequently, we recover from Theorem 2.5 the earlier mentioned theorem of [1] after making the extra assumption that the base field k is algebraically closed.

Let G be an orthogonal or a symplectic group. So a principal G-bundle E_G corresponds to a vector bundle E over M whose fibers are equipped with a nondegenerate bilinear form. This bilinear form on the fibers is symmetric or anti-symmetric depending on whether G is an orthogonal group or a symplectic group. This bilinear form identifies the vector bundle E with its dual E^*. A reduction of structure group of the principal G-bundle E_G to a Levi subgroup of G corresponds to a decomposition

$$E = \bigoplus_{i=1}^{m} E_i$$

of the vector bundle E such that if

$$E^* = \bigoplus_{i=1}^{m} E_i^*$$

is the dual decomposition, then $E_i = E^*_{m-i+1}$, $i \in [1, m]$, with respect to the above mentioned identification of E with E^*. Using this we may reformulate Theorem 2.5 for orthogonal and symplectic groups.

3. Equivariant Higgs bundles

Let Γ be an abstract group and $\Gamma \longrightarrow \mathrm{Aut}(M)$ a homomorphism. So Γ has a left action on the variety M acting as algebraic automorphisms. Let E_G be a principal G-bundle over M equipped with a lift of the action of Γ on M. So Γ has a left action on the variety E_G acting as algebraic automorphisms such that

- the actions of G and Γ on E_G commute;
- the action of Γ on E_G descends to the given action of Γ on M.

Fix a vector bundle F over M equipped with a lift of the action of Γ on M. Let

$$\theta \in H^0(M, \mathrm{ad}(E_G) \otimes F)^{\Gamma} \subset H^0(M, \mathrm{ad}(E_G) \otimes F)$$

be a section left invariant by the action of Γ, where $\mathrm{ad}(E_G) = (E_G \times \mathfrak{g})/G$ is the adjoint vector bundle. We recall that $\mathrm{ad}(E_G)$ is the quotient of $E_G \times \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G, for the following action of G: the action of any $g \in G$ sends $(z, v) \in E_G \times \mathfrak{g}$ to $(zg, \mathrm{Ad}(g^{-1})(v))$. So $\mathrm{ad}(E_G)$ is the Lie algebra bundle for the group scheme $\mathrm{Ad}(E_G)$ over M.

Consider the (reduced) subgroup

$$\mathrm{Aut_r}(E_G, \theta)_{\Gamma} \subset \mathrm{Aut_r}(E_G)$$

defined by all automorphisms that take the section θ to itself and also commute with the action of Γ on E_G.

Take a semisimple element γ as in (3) satisfying the further condition that $\gamma \in \mathrm{Aut_r}(E_G, \theta)_{\Gamma}$.

Proposition 3.1. *For the reduction of structure group $E_{\mathscr{R}} \subset E_G$ constructed in Proposition* 2.2 *for the above $\gamma \in \mathrm{Aut_r}(E_G, \theta)_{\Gamma}$, the following are valid*:

(1) *the action of Γ on E_G leaves the subvariety $E_{\mathscr{R}}$ invariant*;
(2) *the section θ is contained in the subspace*

$$H^0(M, \mathrm{ad}(E_{\mathscr{R}}) \otimes F) \longrightarrow H^0(M, \mathrm{ad}(E_G) \otimes F),$$

where $\mathrm{ad}(E_{\mathscr{R}})$ *is the adjoint bundle of* $E_{\mathscr{R}}$.

Proof. That the action of Γ on E_G leaves $E_{\mathscr{R}}$ invariant follows from the construction of the reduction $E_{\mathscr{R}}$ and the assumption that the automorphism γ commutes with the action of Γ on E_G.

For any closed point $x \in M$, the subalgebra $\mathrm{ad}(E_{\mathscr{R}})_x \subset \mathrm{ad}(E_G)_x$ coincides with the eigenspace corresponding to the eigenvalue 1 of the adjoint action of $\psi_x(\gamma) \in \mathrm{Ad}(E_G)_x$ (the map ψ_x is defined in (2)) on the Lie algebra $\mathrm{ad}(E_G)_x$. Since γ takes θ to itself, it follows therefore that for any dual element $v \in F_x^*$, the element $v(\theta(x)) \in \mathrm{ad}(E_G)$ (the evaluation of the form v) is contained in the subspace $\mathrm{ad}(E_{\mathscr{R}})_x \subset \mathrm{ad}(E_G)_x$. This proves the second part, and the proof of the proposition is complete. $\square$

If we replace $\mathscr{R}$ by $\mathscr{R}' = \alpha^{-1}\mathscr{R}\alpha$, where $\alpha \in G$ is a fixed element, then

$$\mathrm{ad}(E_{\mathscr{R}}) = \mathrm{ad}(E_{\mathscr{R}'}) \subset \mathrm{ad}(E_G),$$

where $E_{\mathscr{R}'} = E_{\mathscr{R}}\alpha$. Therefore, Remark 2.3 remains valid in the present context.

Let

$$\mathrm{Aut}_{\mathrm{r}}^0(E_G, \theta)_\Gamma \subset \mathrm{Aut}_{\mathrm{r}}(E_G, \theta)_\Gamma$$

be the connected component containing the identity element. Take a maximal torus $\mathscr{T}' \subset \mathrm{Aut}_{\mathrm{r}}^0(E_G, \theta)_\Gamma$ and an element

$$\gamma \in \mathscr{T}'$$

such that the centralizer of $\psi_x(\gamma)$ in $\mathrm{Ad}(E_G)_x$ coincides with the centralizer of $\psi_x(\mathscr{T}')$, where ψ_x is defined in (2). Let $\mathscr{R}(\Gamma, \theta) \subset G$ be a subgroup constructed as in (8) for this choice of γ.

Now we have the following analog of Theorem 2.5:

Theorem 3.2. *The reduction of structure group of E_G to the Levi subgroup $\mathscr{R}(\Gamma, \theta)$ constructed in Proposition* 3.1 *for the above element γ is unique in the sense described in Theorem* 2.5. *Furthermore, this reduction of structure group has the property that there is no proper Levi subgroup of $\mathscr{R}(\Gamma, \theta)$ and a reduction of structure group of E_G to it satisfying the two assertions in Proposition* 3.1.

Remark 3.3. Note that the principal Higgs bundles as well as the ramified *G*-bundles (see [4]) form special cases of objects considered in Theorem 3.2.

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